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(August 22, 2000)

Assuming the space dimension is not constant but decreases during the expansion of the Universe, we study chaotic inflation with the potential $m^2\phi^2/2$. Our investigations are based on a model Universe with variable space dimensions. We write down field equations in the slow-roll approximation and define slow-roll parameters by assuming the space dimension to be a dynamical parameter. The dynamical character of the space dimension shifts the initial and final value of the inflaton field to larger values, producing delayed chaotic inflation. We obtain an upper limit for the space dimension at the Planck length. This result is in agreement with previous works on the effective time variation of the Newtonian gravitational constant in a model Universe with variable space dimensions. We present some cosmological consequences and calculate observable quantities including the spectral indices, their scale-dependence, and the mass of the inflaton field. The dynamical behavior of the inflaton field, the variable space dimensions, the scale factor, and their interdependence during the inflationary epoch are also studied.

PACS number(s): 98.80.Cq, 98.80.Hw, 04.50.+h, 98.80.Bp

Preprint : IPM/P-2000/022

I. INTRODUCTION

In recent works [1,2], the wave function and the dynamics of a model Universe with variable space dimension have been studied in detail. In this model, the space dimension decreases during the expansion of the Universe. There is a dimensional constraint that relates the space dimension to the size of the Universe. Here we apply this constraint to a simple model of chaotic inflation and investigate the dynamical behavior of the inflaton field, the scale factor and also the space dimension.

Our motivation for studying chaotic inflation with variable space dimensions is that it enables us to calculate some cosmological quantities and compare them with recent observational data, including the spectral indices. These comparisons indicate to what degree our model Universe with variable space dimensions is consistent with the latest observational data.

Inflationary cosmology has been studied in higher dimensional models; for a recent work see Ref. [3]. Cosmological scenarios of the new inflationary type form a self-similar fractal of dimension slightly less than three [4]. Inflation with fluctuating dimensions describes the Universe as an exponentially large number of independent inflationary domains (mini-Universes) of different dimension [5]. Inflation can describe the transition from higher dimensions to the ordinary three-space and has been studied as a dynamical effect of higher dimensions [6].

In this paper, we study chaotic inflation in a framework of a model Universe with variable space dimensions. This new investigation is important as a conceivable application of the idea of variable space dimensions and as a powerful cosmological test of the model Universe with variable space dimensions.

Our main results in this paper are: *i*) an upper limit for the space dimension at the Planck length, *ii*) a small shift in the value of ϕ_i and ϕ_f to larger values, producing delayed chaotic inflation and *iii*) derivation of the spectral indices and other observable quantities. As we shall see, in this model the number of space dimensions at the Planck length must be less than 10, which is in agreement with the previous result in [2].

After this work was substantially complete we learned the picture of delayed recombination at redshift $z_{rec} \sim 1000$ caused by early sources of Ly α radiation [7]. One possibility is that this delayed recombination may be related to our results of delayed chaotic inflation. But it is good science to bear in mind the possibility that Nature is more complicated than our ideas, a rule that has particular force in cosmology because of the limited empirical basis. Future

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analyses of BOOMERANG data contains estimates of cosmological parameters and informations about the contents and history of the Universe [8]. If BOOMERANG data does not agree with delayed chaotic inflation, it would convincingly rule out our model for delayed inflation. Work on this issue is in progress. We will use a natural unit system that sets k_B, c and $\hbar$ all equal to 1, so that $l_P = M_P^{-1} = \sqrt{G}$.

The paper is organized as follows. In Sec. II we review chaotic inflation and its cosmological consequences in three-space. In Sec. III, chaotic inflation is formulated in any arbitrary constant space dimension. In Sec. IV, we explain the idea of variable space dimension and review the previous works on model Universes with variable space dimension. In Sec. V, we investigate chaotic inflation with variable space dimension and discuss its cosmological consequences in detail. Sec. VI contains our conclusions.

II. BRIEF REVIEW OF INFLATIONARY COSMOLOGY

Despite the success of the standard big bang cosmology in interpreting the Hubble expansion law, the CMBR, and the abundance of light elements, the standard cosmology had some problems before the 1980s [9–11]. Here we shall briefly review some of these problems:

(1) The horizon problem: Using the standard cosmology, in the decoupling epoch with redshift $z_{dec} \sim 1100$, the angular size of the Hubble radius is [12]

$$\theta_H \approx 0.87^o \Omega^{1/2} \left(\frac{z_{dec}}{1100}\right)^{-1/2} \simeq 1^o. \tag{1}$$

Therefore, the angular size of the causality in the CMBR is obtained to within one degree. In contrast, the observations of COBE show that the CMBR is homogeneous on scales larger than one degree to an accuracy of better than one part in 10^4.

(2) The flatness problem: Our Universe is almost flat; its density is within one order of magnitude of the critical density ρ_c. In order to have $\Omega \equiv \rho/\rho_c \sim 1$ today we must assume that Ω was fine-tuned to within $|1-\Omega| \leq 10^{-52}$ at the GUT epoch or $|1-\Omega| \leq 10^{-58}$ at the Planck time t_P. The flatness problem is therefore a problem of understanding why the initial conditions corresponded to a Universe that was so close to spatial flatness. This problem of fine-tuning of the initial density of the Universe feels strange to us [13].

(3) The monopole problem: In the context of grand unified theories, cosmologists expected that there must be a huge number of magnetic monopoles. These monopoles are produced at the grand unification phase transition, where the GUT Higgs fields acquire their nonzero values. The rapidity of the phase transition implies that the correlation length of the Higgs fields is very short, and the fields therefore become tangled in a high density of knots; these knots have the physical properties of superheavy ($\simeq 10^{16}$GeV) magnetic monopoles [14]. For typical grand unified theories, the expected mass density of these magnetic monopoles would exceed the mass density of everything else by a factor of about 10^{12} [15].

(4) The density fluctuation problem: It is now generally believed that galaxies and clusters of galaxies evolved by gravitational instability from small density fluctuations in the early Universe. The required magnitude of fluctuations on galactic scales at the Planck epoch is $\delta\rho/\rho \sim 10^{-56}$ (assuming they are adiabatic). The origin of these tiny fluctuations is mysterious in the standard cosmology and they simply have to be postulated [13].

All these problems (and others we have not listed here) have been either completely resolved or considerably relaxed in the context of one comparatively simple scenario of the evolution of the Universe - the inflationary scenario. For a technical review of inflationary cosmology see Refs. [16–24]. We assume that there is a time interval beginning at t_i and ending at t_f (the "reheating time") during which the Universe is exponentially or quasi-exponentially expanding. The Universe is described by the "de Sitter" or "Inflationary" cosmology during such a period. Now, the key question is that how to obtain inflation. From the Friedmann-Robertson-Walker (FRW) equation it follows that in order to get an exponential increase of the scale factor, the equation of state must be

$$p = -\rho \tag{2}$$

which is not compatible with the standard cosmological model description of matter as a classical ideal gas. As we know, in the early Universe, matter must be described by quantum field theory. The initial hope of the inflationary Universe was that the Higgs field required for gauge symmetry breaking in "grand unified" models would serve as the *inflaton*. The contribution of the scalar field to the energy density ρ and pressure p are

$$\rho(\phi) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}a^{-2}(\nabla\phi)^2 + V(\phi), \tag{3}$$

$$p(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}a^{-2}(\nabla\phi)^2 - V(\phi), \tag{4}$$

where overdot is time derivative and $V(\phi)$ is the potential of scalar field. The crucial equations describing the expansion of the Universe are

$$H^2 = \frac{8\pi\rho}{3M_P^2} - \frac{k}{a^2} \quad \text{Friedmann equation,} \tag{5}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \text{Fluid equation,} \tag{6}$$

where $H = \dot{a}/a$ is the Hubble parameter and the constant k measures the spatial curvature, with k negative, zero and positive corresponding to open, flat, and closed Universe respectively. Substituting Eq.(3) and (4) into Friedmann and fluid equation, we are led to

$$H^2 = \frac{8\pi}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}a^{-2}(\nabla\phi)^2 + V(\phi)\right] - \frac{k}{a^2}, \tag{7}$$

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^2\phi = -V'(\phi), \tag{8}$$

where prime indicates $d/d\phi$. Most of the current realizations of potential-deriven inflation are based on satisfying the conditions[1]

$$\dot{\phi}^2\ ,\ a^{-2}(\nabla\phi)^2 \ll V(\phi), \tag{9}$$

$$\ddot{\phi} \ll 3H\dot{\phi}, \tag{10}$$

$$-\dot{H} \ll H^2, \tag{11}$$

via the slow-rolling assumption [10,11]. The basic condition $\dot{\phi}^2 \ll V(\phi)$ can now be written by the slow-roll relation as

$$\epsilon \equiv \frac{M_P^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1. \tag{12}$$

Also, we can differentiate this expression to obtain the criterion $V'' \ll V'/M_P$. Using slow-roll once more gives $3H\dot{\phi}/M_P$ for the RHS, which is in turn $\ll 3H\sqrt{V}/M_P$ because $\dot{\phi}^2 \ll V$, giving finally

$$\eta \equiv \frac{M_P^2}{8\pi}\left(\frac{V''}{V}\right) \ll 1. \tag{13}$$

The necessary conditions for the slow-roll approximation to hold are

$$\epsilon \ll 1 \quad , \quad |\eta| \ll 1. \tag{14}$$

Although these are necessary conditions for the slow-roll approximation to hold, they are not sufficient [20]. The slow-rolling assumes that a term can be neglected in each of the equations of motion, leaving the simpler set for a homogeneous scalar field

$$H^2 \simeq \frac{8\pi}{3M_P^2}V(\phi), \tag{15}$$

$$3H\dot{\phi} \simeq -V'(\phi). \tag{16}$$

In Eq. (15), we have ignored the curvature term k, since we know that it will quickly become negligible once inflation starts. The relation $|\eta| \ll 1$ can now be written by Eq.(16) as $|V''| \ll H^2$. Successful inflation in any inflationary model requires more than 60 e-folding of the expansion. The implications of this are easily calculated using the slow-rolling equation, which gives the number of e-folding between ϕ_i and ϕ_f as

$$\mathcal{N} = \int_{t_i}^{t_f} H(t)dt = \ln(a_f/a_i) \simeq -\frac{8\pi}{M_P^2}\int_{\phi_i}^{\phi_f} \frac{V}{V'}\, d\phi, \tag{17}$$

[1]Note that by differentiating the slow-roll condition $a^{-2}(\nabla\phi)^2 \ll V(\phi)$ with respect to the space coordinate, one obtains $a^{-2}\nabla^2\phi \ll V'(\phi)$. Therefore, the third term on the LHS of Eq. (8) can be neglected in comparison with the term on the RHS of (8).

where i and f refer to the beginning and end of the inflationary epoch. Let us now consider how inflation solves the problems of the standard cosmology.

(1) Solution of the horizon problem: The comoving size of the region in the last scattering surface from which we receive signals today is

$$\ell(t_0, t_{dec}) = \int_{t_{dec}}^{t_0} \frac{dt}{a(t)}, \tag{18}$$

where comoving size of the particle horizon at decoupling will be

$$\ell(t_{dec}, 0) = \int_0^{t_{dec}} \frac{dt}{a(t)}. \tag{19}$$

To solve the horizon problem, inflation brings the entire observed region of the last scattering surface into a causally connected patch, so that

$$\Re = \frac{\ell(t_{dec}, 0)}{\ell(t_0, t_{dec})} \cong 4 \times 10^4, \tag{20}$$

Therefore light can travel much further before decoupling than it can afterwards. This can not be done with standard evolution, but can be achieved by inflation. In the standard cosmology, we have $\Re \ll 1$.

(2) Solution of the flatness problem: Using the definition of the critical density

$$\rho_c \equiv \frac{3M_P^2 H^2}{8\pi}, \tag{21}$$

and Eq.(5), we are led to

$$1 - \Omega^{-1} = \frac{3M_P^2 k}{8\pi\rho a^2}. \tag{22}$$

This expression implies that the quantity $(\Omega^{-1} - 1)\rho a^2$ is conserved for an arbitrary equation of state. It follows, therefore, that

$$(\Omega_i^{-1} - 1)\rho_i a_i^2 = (\Omega_f^{-1} - 1)\rho_f a_f^2, \tag{23}$$

and, if we assume that the standard big bang model is valid for $t > t_f$, we may deduce that [30]

$$\rho_i a_i^2 \left|\Omega_i^{-1} - 1\right| \approx 10^{-51} \rho_f a_f^2 \left|\Omega_0^{-1} - 1\right|. \tag{24}$$

Assuming the initial conditions are not fine tuned (i.e. $\Omega_i \simeq \mathcal{O}(1)$), and Ω of order unity today (i.e. $\Omega_0 \simeq \mathcal{O}(1)$) we have $|1 - \Omega_f| \lesssim 10^{-52}$ at the GUT epoch, and so $\mathcal{N} \gtrsim 60$ e-foldings of expansion are needed [27,28]. (It is easily shown that $|1 - \Omega| \simeq |1 - \Omega^{-1}|$ because $|1 - \Omega|$ is a very small number).

(3) Solution of the monopole problem: Monopoles are point-like topological defects that would be expected to arise in any phase transition near the GUT scale ($t \sim 10^{-35}$). If they form at a rate of approximately one per horizon volume at this time, then it follows that the present Universe would contain $\Omega \gg 1$ in monopoles. This unpleasant conclusion is avoided if inflation can make the horizon much larger than its classical value. The GUT fields have then been aligned over a vast scale, so topological defect formation becomes extremely rare. Their density becomes negligible very quickly because at the end of inflation, the energy density of the Universe can be turned into conventional matter without recreating the unwanted relics. This can be achieved by ensuring that during the conversion, known as reheating, the temperature never gets hot enough again to allow their thermal reaction to proceed. Such successful reheating allows us to get back into the hot big bang Universe, recovering all its later successes such as nucleosynthesis and the CMBR.

(4) Solution of the density fluctuation problem: Most importantly, inflation provides a causal mechanism for generating the primordial perturbation required for galaxies, clusters and even larger objects. During inflation the scale factor grows quasi-exponentially, while the Hubble radius remains almost constant. Consequently the wavelength of a quantum fluctuation soon exceeds the Hubble radius. The amplitude of the fluctuation therefore becomes 'frozen'. Once inflation has ended, however, the Hubble radius increases faster than the scale factor, so the fluctuations eventually reenter the Hubble radius during the radiation or matter dominated eras. The fluctuations that exit around 60 e-folding or so before reheating reenter with physical wavelengths in the range accessible to cosmological observations. At a galaxy scale ($\sim 10^{21} - 10^{22}$cm), inflation gives a desirable amplitude $\delta\rho/\rho \sim 10^{-4} - 10^{-5}$ which is consistent with the measurement of CMBR anisotropies by the COBE satellite on scales above 10^o [31].

A. Chaotic inflation

1. Field equations

Historically, there were several different versions of the inflationary Universe scenario. At present, it seems that the simplest and most general one is chaotic inflation with a quadratic potential of a massive scalar field

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{25}$$

In this model, it is assumed that as part of an initially chaotic state, $\phi \gtrsim M_P$ or even $\phi \gg M_P$. It is argued that so long as $V(\phi) \leq M_P^4$, these initial conditions are justified. The scalar field ϕ satisfies $V(\phi) \lesssim M_P^4$ at the Planck time as well. Let us consider a domain of the Universe with an initial size on the order of the Planck length $l_P \sim M_P^{-1} \sim H^{-1}$, where H^{-1} is the horizon radius in the inflationary Universe. If the classical field ϕ is sufficiently homogeneous in this domain, then its behavior inside this domain does not depend on the physical processes outside the horizon. Inserting the potential (25) in Eqs. (15) and (16), we obtain

$$H^2 \simeq \frac{4\pi}{3M_P^2}m^2\phi^2, \tag{26}$$

$$3H\dot{\phi} \simeq -m^2\phi. \tag{27}$$

In that case, any solution rapidly approaches the regime

$$\phi(t) = \phi_i - \frac{mM_P}{2\sqrt{3\pi}}t, \tag{28}$$

$$a(t) = a_i \exp\left(\frac{2\pi}{M_P^2}\left[\phi_i^2 - \phi^2(t)\right]\right). \tag{29}$$

According to Eqs. (28) and (29), during the interval time $\Delta t \leq \phi/(mM_P)$ the value of the field ϕ remains almost unchanged. The inequality (11) means that during the time interval $\Delta t \gg H^{-1}$ the value of the Hubble constant H remains almost unchanged and the Universe expands quasi-exponentially

$$a(t) = a_i \exp\left(Ht\right), \tag{30}$$

where the Hubble "constant" is given by

$$H \approx \sqrt{\frac{4\pi}{3}}\frac{m\phi}{M_P}, \tag{31}$$

and the slow-roll parameters are

$$\epsilon = \eta = \frac{1}{4\pi}\frac{M_P^2}{\phi^2}. \tag{32}$$

So inflation can proceed provided $|\phi| > M_P/\sqrt{4\pi}$, i.e. as long as we are not to close to the minimum. When the field ϕ becomes smaller than about $M_P/\sqrt{4\pi}$, the friction term in equation (8) becomes small, and ϕ oscillates rapidly near its equilibrium value of zero. So the regime of quasi-exponential expansion occurs for $\phi \geq M_P/\sqrt{4\pi}$. For $\phi \leq M_P/\sqrt{4\pi}$ the field ϕ oscillates rapidly and if this field interacts with other fields, these oscillations lead to abundant particle production. Its potential energy $m^2\phi^2/2$ is transformed into heat.

Substituting Eq.(25) into (17), one can then calculate the number of e-foldings

$$\mathcal{N} = -\frac{4\pi}{M_P^2}\int_{\phi_i}^{\phi_f = M_P/\sqrt{4\pi}} \phi\, d\phi = 2\pi\frac{{\phi_i}^2}{M_P^2} - \frac{1}{2}. \tag{33}$$

In this free-field model $\mathcal{N}$ depends only on ϕ_i and not on the inflaton mass m. Thus the number of e-foldings will exceed 60 provided that

$$\phi_i > \sqrt{\frac{60}{2\pi}}M_P \approx 3.10M_P. \tag{34}$$

Although the value of the scalar field is larger than M_P, the energy density can be low compared to the Planck scale

$$V(\phi_i) = \frac{1}{2}m^2{\phi_i}^2 > \frac{15}{\pi}{M_P}^2 m^2. \tag{35}$$

For $m < \mathcal{O}(10^{-5})M_P$, which is necessary to produce density perturbations $\delta\rho/\rho \lesssim 10^{-4}$, the potential energy density is greater than $\sim 5 \times 10^{-10} M_P^4$. Since it is presumably the energy density that is relevant to gravity, one does not expect this situation to lead to strong quantum gravity effects. Although a suitable amount of e-folding did not put a strong constraint on chaotic inflation, a stronger constraint is derivable by considering density perturbations. The conditions that $V(\phi_i) \lesssim M_P^4$ and $m \sim 10^{-6}M_P$ now imply that $\phi_i \lesssim M_P^2/m \simeq 10^6 M_P$.

During inflation the Universe becomes divided into many exponentially large domains or mini-Universes, inside which the properties of elementary particles and even the dimensionality of space-time may be different. In this scenario, the Universe looks like a self-reproducing space-time foam consisting of exponentially large inflationary bubbles. In contrast with a one-bubble Friedmann Universe [32], in this picture the whole Universe looks like a cluster of mini-Universes with some of them inflationary. Let us consider for definitness a closed Universe of a typical initial size $\mathcal{O}(l_P)$. A typical initial value ϕ inside these mini-Universes would be M_P^2/m. From Eq.(29) we see that by the time ϕ approaches zero, the Universe has expanded by an inflating factor

$$l \sim l_P \exp\left(\frac{2\pi\phi_i^2}{M_P^2}\right) \sim l_P \exp\left(\frac{2\pi M_P^2}{m^2}\right). \tag{36}$$

For m on the order of $10^{-6}M_P$, this implies that the inflationary domains of the Universe typically expand to

$$l \sim 10^{-33} \exp\left(2\pi 10^{12}\right) > 10^{10^{12}}\mathrm{cm}, \tag{37}$$

which is much greater than the size of the observable Universe $\sim 10^{28}$cm. After such a large inflation the term k/a^2 in Eq. (7) becomes negligiblely small compared with H^2, which means that the Universe becomes flat and its geometry locally Euclidean. This implies that the total density of the Universe ρ becomes almost exactly equal to the critical density and $\Omega \sim 1$. For similar reasons the Universe becomes locally homogeneous and isotropic. The density of all "undesirable" object (monopoles, domain walls, gravitons) created before or during inflation becomes exponentially small, and they never appear again if the reheating temperature, T_R, is not too large.

Inflation ends when $t = t_f$. The value of t_f can be obtained by Eq. (28)

$$t_f = \frac{2\sqrt{3\pi}\,(\phi_i - \phi_f)}{mM_P} \simeq 7.71 \times 10^{-37}\,\mathrm{sec}, \tag{38}$$

where we take $m \simeq 1.21 \times 10^{-6}\mathrm{M}_P$, see below.

2. Observational consequences

Starting with a Universe which is absolutely homogeneous and isotropic at the classical level, the inflationary expansion of the Universe will "freeze in" the vacuum fluctuations of the inflaton field, so that they become essentially classical quantities. On each comoving scale, this happens soon after the horizon exit. The typical comoving scale of causal correlation or the comoving Hubble length, $1/\dot{a}$, decreases during inflation because $\ddot{a} > 0$. A comoving scale $1/k$ is said to leave the horizon when $1/k = 1/\dot{a}$ or $k = aH$. Associated with these vacuum fluctuations are primordial energy density perturbations, which survive after inflation and may be the origin of all structure in the Universe. In particular, it may be responsible for the observed CMBR anisotropy and for the large scale distribution of galaxies and dark matter. Inflation also generates primordial gravitational waves as vacuum fluctuations which may contribute to the low multipoles of the CMBR anisotropy.

When it was first proposed in 1982, this remarkable paradigm received comparatively little attention. For one thing observational tests were weak and for another the inflationary density perturbation was not the only candidate for the origin of structure. In particular, the situation changed dramatically in 1992, when COBE measured the CMBR anisotropy on large angular scales [31] and another dramatic change is now in progress with the advent of smaller scale measurement. Subject to confirmation by the latter, it seems that the paradigm of slow-roll inflation is the only one not in conflict with observation [33]. Exact results for the spectra are not known for arbitrary potentials $V(\phi)$. These can be calculated analytically via the slow-roll approximation. The accuracy required depends on the way in which the results shall be used; for normalizing to COBE it is valid to use the well-known lowest-order results [27], which give the density perturbation (i.e., scalar) spectrum $A_S(k)$ and gravitational wave (i.e., tensor) spectrum $A_T(k)$ as

$$A_S^2(k) = \frac{512\pi}{75} \frac{V^3}{M_P^6 V'^2}\bigg|_{k=aH}, \tag{39}$$

$$A_T^2(k) = \frac{32}{75} \frac{V}{M_P^4}\bigg|_{k=aH}. \tag{40}$$

Here k is the comoving wavenumber and so k/a is the physical wavenumber.

The primordial density perturbation is gaussian, in other words its Fourier components δ_k are uncorrelated and have random phase. Its spectrum $A_S(k)$, defined roughly as the expectation value of $|\delta_k|^2$ at the epoch of horizon exit, defines all of its stochastic properties. The shape of the spectra are conveniently defined by the spectral indices in terms of the slow-roll parameters

$$n(k) - 1 \equiv \frac{d \ln A_S^2(k)}{d \ln k}\bigg|_{k=aH} = -6\epsilon + 2\eta, \tag{41}$$

$$n_T(k) \equiv \frac{d \ln A_T^2(k)}{d \ln k}\bigg|_{k=aH} = -2\epsilon. \tag{42}$$

In the context of the cold non-baryonic dark matter, on observational constraint on the spectral index n has been obtained. If n is practically scale-independent, as predicted by most models of inflation, present data requires $n \simeq 1.0 \pm 0.1$ [29]. The predictions of different models for the spectral index n and for its scale dependence are given in [33]. The ratio of the two spectra

$$\frac{A_T^2}{A_S^2}\bigg|_{k=aH} = -\frac{n_T}{2}, \tag{43}$$

is not independent of the tensor spectral index. The tensor spectral index n_T, which is the hardest thing to directly observe, can be eliminated through its relation to the ratio A_T^2/A_S^2. To indicate the relative importance of the gravitational waves, we define a quantity r by [27,34]

$$r = 12.4 \frac{A_T^2(k)}{A_S^2(k)}\bigg|_{k=aH}, \tag{44}$$

which measures, in the matter-dominated and Sachs-Wolfe approximation, the relative importance of gravitational waves and density perturbations in contributing to the relevant microwave multipole.

The appropriate point on the inflationary potential at which to evaluate the spectra is given by $\mathcal{N}_\star$, which is not specified very accurately. Normally it is a perfectly fine approximation to say that the scales of interest to us crossed outside the Hubble radius 60 e-foldings before the end of inflation. Then the e-folding formula

$$\mathcal{N}(\phi) \simeq -\frac{8\pi}{M_P^2} \int_\phi^{\phi_f} \frac{V}{V'}\, d\phi, \tag{45}$$

tells us the value of $\phi_\star$. ϕ_f could be calculated numerically, but is normally given to adequate accuracy by the breakdown of the slow-roll conditions, taken as $\epsilon_f = \eta_f = 1$. Having located $\phi_\star$, one immediately gets the slow-roll parameters and hence the spectral indices at that scale. We can differentiate Eq. (45) to obtain the relation between $\phi_\star$ and when a physical scale $a/k_\star$ crosses the Hubble radius H^{-1}, *i.e.* $k_\star = aH$,

$$\frac{d \ln k}{d\phi} = -\frac{8\pi}{M_P^2} \frac{V}{V'}, \tag{46}$$

since within the slow-roll approximation $k \simeq \exp(-\mathcal{N})$. Using this equation and Eqs. (41) and (42), the scale-dependence of the spectral indices are easily shown [35]

$$\frac{dn(k)}{d \ln k}\bigg|_{k_\star} \equiv -24\epsilon_\star^2 + 16\epsilon_\star \eta_\star - \frac{M_P^4}{32\pi} \frac{V'V'''}{V^2}\bigg|_{\phi_\star}, \tag{47}$$

$$\frac{dn_T(k)}{d \ln k}\bigg|_{k_\star} \equiv 4\epsilon_\star \eta_\star - 8\epsilon_\star^2. \tag{48}$$

In chaotic inflation, with the potential $m^2\phi^2/2$ for which the slow-roll parameters are given in Eq.(32), inflation ends when $\phi_f = M_P/\sqrt{4\pi}$. For definitness, we take $\mathcal{N}_\star = 60$ and from Eq. (45) we find $\phi_\star = 3.10\, M_P$. So for this model one predicts

$$A_S(k_\star) \simeq 15.735 \frac{m}{M_P} \quad , \quad A_T(k_\star) \simeq 1.492 \frac{m}{M_P}, \tag{49}$$

and

$$n(k_\star) = 0.967 \quad , \quad n_T(k_\star) = -1.657 \times 10^{-2} \quad , \quad r(k_\star) = 1.027 \times 10^{-1}. \tag{50}$$

We specify the COBE normalization of the density perturbations, following Ref. [36], at the present Hubble scale $k = a_0 H_0$, and define

$$\delta_H \equiv A_S(a_0 H_0). \tag{51}$$

Assuming the gravitational waves are negligible, $A_T \ll A_S$, the COBE result requires [36]

$$\delta_H = 1.91 \times 10^{-5}. \tag{52}$$

Using Eqs. (12), (39) and (52), it is shown that when the observable Universe leaves the horizon the inflationary energy scale is given by

$$V_\star^{1/4} = 6.60\, \epsilon_\star^{1/4} \times 10^{16} \mathrm{GeV}. \tag{53}$$

As always, a star denotes the epoch of horizon exit $k_\star \equiv aH$. Taking $\mathcal{N}_\star = 60$ and $\phi_\star = 3.10 M_P$ from Eq. (32) we have $\epsilon_\star = 8.258 \times 10^{-3}$ and the inflationary energy scale is

$$V_\star^{1/4} \simeq 1.99 \times 10^{16} \mathrm{GeV}. \tag{54}$$

In most model of inflation the potential at the end of inflation is nearly equal to $V_\star$. From Eqs.(49) and (52), one gets $m \simeq 1.21 \times 10^{-6} M_P$. Using Eqs. (47) and (48), the scale-dependence of the spectral indices are

$$\left.\frac{dn(k)}{d\ln k}\right|_{k_\star} \simeq -5.489 \times 10^{-4}, \tag{55}$$

and

$$\left.\frac{dn_T(k)}{d\ln k}\right|_{k_\star} \simeq -2.745 \times 10^{-4}. \tag{56}$$

Rather than use Eq.(43) to set the relative normalization of scalars and tensors, one may use the more accurate expression by fitting to COBE data [37,38]

$$\left.\frac{A_T^2(k)}{A_S^2(k)}\right|_{k=aH} = -\frac{n_T}{2}\left(1 - \frac{n_T}{2} + (n-1)\right). \tag{57}$$

Using this, r is related to the spectral indices by

$$r = -6.2\, n_T \left(1 - \frac{n_T}{2} + (n-1)\right). \tag{58}$$

According to an accurate calculation, the relative contribution of gravitational waves to the COBE anisotropy is actually $0.75r$, which reduces the deduced value of δ_H by a factor $\simeq (1+0.75r)^{-1/2}$ when compared with Eq. (52). A fitting function for the four-year COBE normalization is accurately represented by [36]

$$\delta_H(n,r) = 1.91 \times 10^{-5} \frac{\exp[1.01(1-n)]}{\sqrt{1+0.75\, r}}. \tag{59}$$

What about the future? Large-scale structure observations and especially microwave background observations can strongly discriminate between inflationary models. When they are made, the magnificent COBE normalization will be improved and most existing inflation models will be ruled out. Much more interesting is the situation with the spectral index. The Planck satellite will probably measure $n(k)$ with an accuracy of order $\Delta n \sim 0.01$, which as already mentioned will be a powerful discriminator between models of inflation. The Planck satellite probes a range $\Delta \ln k \simeq 6$ and will measure the scale-dependence $dn/d\ln k$ if it is bigger than a few times 10^{-3} [33].

III. INFLATIONARY COSMOLOGY WITH CONSTANT SPACE DIMENSION

Assuming the space dimension is constant and has an arbitrary value, we now formulate the inflationary cosmology in a constant D-space. The crucial equations are given by [1,2]

$$H^2 = \frac{16\pi\rho}{D(D-1)M_P^2} - \frac{k}{a^2} \qquad \text{Friedmann equation,} \tag{60}$$

$$\dot{\rho} + DH(\rho + p) = 0 \qquad \text{Fluid equation.} \tag{61}$$

Inserting $k = 0$ and the energy density and pressure of a homogeneous inflaton field

$$\rho \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{62}$$

$$p \equiv \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{63}$$

in Eqs. (60) and (61), we are led to

$$H^2 = \frac{16\pi}{D(D-1)M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{64}$$

$$\ddot{\phi} + DH\dot{\phi} = -V'(\phi). \tag{65}$$

In a constant D-space the slow-roll conditions are given by

$$\dot{\phi}^2 \ll V(\phi), \quad \ddot{\phi} \ll DH\dot{\phi}, \quad -\dot{H} \ll H^2. \tag{66}$$

Using these conditions, Eq. (64) and (65) can be written in the simpler set

$$H^2 = \frac{16\pi V(\phi)}{D(D-1)M_P^2}, \tag{67}$$

$$DH\dot{\phi} \simeq -V'(\phi). \tag{68}$$

During inflation, H is slowly varying in the sense that its change per Hubble time, $\epsilon \equiv -\dot{H}/H^2$ is less than 1 [34,40]. The slow-roll condition $|\eta| \ll 1$ is actually a consequence of the condition $\epsilon \ll 1$ plus the slow-roll approximation $DH\dot{\phi} \simeq -V'$. Indeed, differentiating Eq. (68) one finds

$$\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \eta, \tag{69}$$

where the slow-roll parameters in any constant space dimension are defined by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{(D-1)M_P^2}{32\pi}\left(\frac{V'}{V}\right)^2, \tag{70}$$

$$\eta \equiv \frac{(D-1)M_P^2}{16\pi}\left(\frac{V''}{V}\right). \tag{71}$$

The number of e-foldings between t_i and t_f is given by

$$\mathcal{N} = \int_{t_i}^{t_f} H(t)dt = \ln\left(\frac{a_f}{a_i}\right) \simeq -\frac{16\pi}{(D-1)M_P^2}\int_{\phi_i}^{\phi_f} \frac{V}{V'}d\phi. \tag{72}$$

For the chaotic inflation with the $m^2\phi^2/2$ potential, the solution of Eqs. (67) and (68) are given by

$$\phi(t) = \phi_i - \frac{mM_P}{2}\sqrt{\frac{D-1}{2\pi D}}\,t, \tag{73}$$

$$a(t) = a_i \exp\left(\frac{4\pi}{M_P^2(D-1)}\left[\phi_i^2 - \phi^2(t)\right]\right). \tag{74}$$

Using the slow-roll parameters

$$\epsilon = \eta = \frac{M_P^2(D-1)}{8\pi\phi^2}, \tag{75}$$

and the failure of the slow-roll conditions

$$\max\{\epsilon_f; |\eta_f|\} \simeq 1, \tag{76}$$

one concludes that $\phi_f = M_P\sqrt{\frac{D-1}{8\pi}}$. Substituting this value of ϕ_f into Eq. (72), one gets

$$\mathcal{N} = \frac{4\pi\phi_i^2}{(D-1)M_P^2} - \frac{1}{2}. \tag{77}$$

Considering the upper bound of the number of e-foldings, $\mathcal{N} > 60$, we obtain

$$\phi_i > \frac{1}{2}\sqrt{\frac{60}{\pi}(D-1)}M_P. \tag{78}$$

From Eq. (74), we see that by the time ϕ approaches zero the Universe has expanded by an inflating factor

$$l \sim l_P \exp\left(\frac{4\pi\phi_i^2}{M_P^2(D-1)}\right) \sim l_P \exp\left(\frac{4\pi M_P^2}{m^2(D-1)}\right), \tag{79}$$

where we use the condition $\phi_i \lesssim M_P^2/m$. For $m \sim 10^{-6}M_P$, this implies that the inflationary domains of the Universe expand to

$$l \sim 10^{-33}\exp\left(\frac{4\pi\times 10^{12}}{D-1}\right). \tag{80}$$

For $1 < D \leq 6$ we have $l > 10^{10^{12}}$. It can be easily shown that for $D = 3$ the equations of this section approach those of Sec. II.

IV. REVIEW OF A MODEL UNIVERSE WITH VARIABLE SPACE DIMENSION

In recent years there have been attempts to study a cosmological model with variable space dimension [25,26,1,2]. Mansouri and Nasseri [1] have studied the model Universe with variable space dimension in the framework of quantum cosmology in detail. Using the appropriate boundary conditions and the semiclassical approximation, they have calculated the wave function and also the tunneling probability density. In the limit of constant space dimension, the wave function of the model does not have a unique behavior. It can either lead to the Hartle-Hawking wave function, a modified Linde wave function, or a more general one. The Vilenkin wave function is excluded. The tunneling probability of the model is always more than that of the de Sitter minisuperspace in three-space as suggested by Vilenkin, Linde, and others. Therefore the creation of the model Universe with variable space dimension is much more probable than that of the de Sitter three-space. In the limit of three-space dimensions, the probability density of our model Universe approaches that of Vilenkin and Linde, being $\exp(-2|S_E|)$, where S_E is the Euclidean action.

The model Universe with variable space dimension indicates that the Vilenkin wave function is not stable with respect to the variation of space dimension. These studies about the wave function of the model Universe with variable space dimension and its tunneling probability are based on new Lagrangian formulations of the model. The details about these new formulations and their consequences on the evolution equation of the space dimension and also on the classical turning points of the model Universe have been studied by Mansouri and Nasseri [1]. Moreover, dynamical behavior of the space dimension effects the Wheeler-DeWitt equation and its potential. The potential of the Wheeler-DeWitt equation in the de Sitter three-space has two turning points, one of them in $a = 0$ and the other in $a = l_P$, where l_P is the Planck length. There is only one potential barrier with a height of the order of the square of the Planck energy $\sim 10^{39}\text{GeV}^2$. In contrast, in the model Universe with variable space dimension, the potential of the Wheeler-DeWitt equation has two potential barriers. The height of the potential barrier with the smaller scale factor generally is less than that of one with the larger scale factor. Moreover, the height of the potential barrier with the larger scale factor is much more than the square of the Planck energy [1]. There is a constraint in this model which can be written as

$$\left(\frac{a}{\delta}\right)^{D} = \left(\frac{a_0}{\delta}\right)^{D_0} = e^{C}, \tag{81}$$

or

$$\frac{1}{D} = \frac{1}{C}\ln\left(\frac{a}{a_0}\right) + \frac{1}{D_0}. \tag{82}$$

Here a is the scale factor of the Friedmann Universe, D the variable space dimension, δ the characteristic minimum length of the model, and C a constant of the model. The zero subscript in any quantity , e.g. in a_0 and D_0, denotes its present value. Note that in constraints (81) and (82), the space dimension is a function of cosmic time t. Time derivative of Eq. (81) leads to

$$\dot{D} = -\frac{D^2\dot{a}}{Ca}. \tag{83}$$

It is easily seen that the case of constant space dimension corresponds to when C tends to infinity. In Ref. [1], some shortcomings of the original model proposed in [25] have been shown, regarding the fields equations and their results. In [1], it has been mentioned that the Lagrangian is not unique. Using the Hawking-Ellis action of a perfect fluid, for the model Universe with variable space dimension, the following Lagrangians have been obtained [1]

$$\begin{aligned} L_I := & -\frac{V_D}{2\kappa N}\left(\frac{a}{a_0}\right)^{D} D(D-1)\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{N^2k}{a^2}\right] \\ & - \rho N V_D\left(\frac{a}{a_0}\right), \end{aligned} \tag{84}$$

and

$$\begin{aligned} L_{II} := & -\frac{V_D}{2\kappa N}\left(\frac{a}{a_0}\right)^{D} \\ & \times\left\{\frac{2\dot{D}\dot{a}}{a} + \frac{2D\dot{a}\dot{D}}{a}\ln\frac{a}{a_0} + D(D-1)\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{N^2k}{a^2}\right] + \frac{2D\dot{D}\dot{a}}{a}\left(\frac{a}{a_0}\right)^{D}\frac{d\ln V_D}{dD}\right\} \\ & - \rho V_D N\left(\frac{a}{a_0}\right)^{D}. \end{aligned} \tag{85}$$

Here $\kappa = 8\pi M_P^{-2}$ and V_D the volume of the space-like sections

$$V_D = \begin{cases} \frac{2\pi^{(D+1)/2}}{\Gamma[(D+1)/2]}, & \text{if } k=+1, \\ \frac{\pi^{(D/2)}}{\Gamma(D/2+1)}{\chi_c}^{D}, & \text{if } k=0, \\ \frac{2\pi^{(D/2)}}{\Gamma(D/2)}f(\chi_c), & \text{if } k=-1, \end{cases} \tag{86}$$

where χ_C is a cut-off and $f(\chi_c)$ is a function thereof (see Ref. [1] and Appendix). In the limit of constant space dimension, L_I and L_{II} approach

$$\begin{aligned} L^0_{I,II} := & -\frac{V_{D_0}}{2\kappa N}\left(\frac{a}{a_0}\right)^{D_0} D_0(D_0-1)\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{N^2k}{a^2}\right] \\ & - \rho N V_{D_0}\left(\frac{a}{a_0}\right)^{D_0}. \end{aligned} \tag{87}$$

Varying the Lagrangian L_I with respect to N and a, we find the following equations of motion in the gauge $N=1$, respectively

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{2\kappa\rho}{D(D-1)}, \tag{88}$$

$$\begin{aligned} & (D-1)\left\{\frac{\ddot{a}}{a} + \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\left(-\frac{D^2}{2C}\frac{d\ln V_D}{dD} - 1 - \frac{D(2D-1)}{2C(D-1)} + \frac{D^2}{2D_0}\right)\right\} \\ & +\kappa p\left(-\frac{d\ln V_D}{dD}\frac{D}{C} - \frac{D}{C}\ln\frac{a}{a_0} + 1\right) = 0. \end{aligned} \tag{89}$$

TABLE I. Values of C and δ for $D_0 = 3$ and $a_0 = H_0^{-1}$.

D_P	C	$\log_{10}(\delta/l_P)$	δ(cm)
3	$+\infty$	$-\infty$	0
4	1678.8	-182.27	8.6×10^{-216}
10	599.57	-26.03	1.5×10^{-59}
25	476.93	-8.28	8.4×10^{-42}
$+\infty$	419.70	0	l_P

Using (83) and (88), the evolution equation of the space dimension can be obtained by

$$\dot{D}^2 = \frac{D^4}{C^2}\left[\frac{2\kappa\rho}{D(D-1)} - k\delta^{-2}e^{-2C/D}\right]. \tag{90}$$

The continuity equation of the model Universe with variable space dimension can be obtained by (88) and (89)

$$\frac{d}{dt}\left[\rho\left(\frac{a}{a_0}\right)^D V_D\right] + p\frac{d}{dt}\left[\left(\frac{a}{a_0}\right)^D V_D\right] = 0. \tag{91}$$

This continuity equation can be integrated for the case of dust or radiation to obtain the energy density as a function of time. For the radiation era, $p = \rho/D$, we have

$$\rho = \rho_{eq} e^{C(D/D_{eq}-1)}\left(\frac{D}{D_{eq}}\right)^{C/D_{eq}}\frac{V_{D_{eq}}}{V_D}\exp\left(-\int_{D_{eq}}^{D} dD\frac{1}{D}\frac{d\ln V_D}{dD}\right), \tag{92}$$

and for the matter era, $p = 0$,

$$\rho = \rho_0 e^{C(D/D_0-1)}\frac{V_{D_0}}{V_D}, \tag{93}$$

where D_{eq} is the space dimension of the Universe when the scale factor is equal to the scale factor of the equality epoch [2]. In the model Universe with variable space dimension, based on Lagrangian L_{II}, field equations have been obtained in Ref. [1]. We define D_P as the space dimension of the Universe when the scale factor is equal to the Planck length l_P. Taking the scale of the Universe at $D_0 = 3$ to be the present value of the Hubble radius H_0^{-1} and the space dimension in the Planck length to be $4, 10$, or 25, from Kaluza-Klein and superstring theories, we can obtain from Eqs. (81) and (82) the corresponding value of C and δ. In Table I, values of C and δ for some interesting values of D_P are given. These values are calculated by assuming $D_0 = 3$ and $H_0^{-1} = 3000h^{-1}\text{Mpc} = 9.2503\times10^{27}h^{-1}\text{cm}$. Since the value of C and δ are not very sensitive to h we take for simplicity $h = 1$.

Time variation of the Newtonian gravitational constant G has been studied in the model Universe with variable space dimension [2]. Using the Lagrangian formulation of this model, the effective gravitational constant has been given as a function of time. To compare it with observational data, a test theory for the time variation of G is formulated. A power law behavior of the time variation of G is proposed. Within this test theory it is possible to restrict the value of C and give an upper bound for the space dimension at the Planck era. It is shown that theories based on L_I and L_{II} are observationally ruled out for $D_P \geq 10$ [2].

V. CHAOTIC INFLATION WITH VARIABLE SPACE DIMENSION

Let us now study the inflationary cosmology when the space dimension is a dynamical parameter. To do this, we first write down our formulation for a general potential and then use our results in the $m^2\phi^2/2$ potential.

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[2]Note that Eq. (70) of Ref. [1] must be corrected by replacing D_{eq} and ρ_{eq} with D_0 and ρ_0, respectively. For the radiation era, we cannot use the present values of the energy density ρ_0 and the space dimension D_0. For the matter era, Eq. (71) of Ref. [1] is correct.

A. General formulations

To study inflation in the framework of the model Universe with variable space dimension, the crucial equations are obtained by substituting Eqs. (62) and (63) into Eqs. (88) and (91). Therefore one finds

$$H^2 = \frac{16\pi}{D(D-1)M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2} \quad \text{Friedmann equation,} \tag{94}$$

$$\ddot{\phi} + DH\dot{\phi} + \dot{D}\dot{\phi}\left(\ln\frac{a}{a_0} + \frac{d\ln V_D}{dD}\right) = -V'(\phi) \quad \text{Fluid equation.} \tag{95}$$

In the case of dynamical space dimension, the slow-roll conditions are the same as in that of constant space dimension, see Eq. (66). It is worth mentioning that in the slow-roll condition $\ddot{\phi} \ll DH\dot{\phi}$, where D is a dynamical parameter. When inflation starts, the curvature term can be neglected, $k = 0$. So, in the slow-roll approximation, Eqs. (94) and (95) lead to

$$H^2 \simeq \frac{16\pi V(\phi)}{D(D-1)M_P^2}, \tag{96}$$

$$DH\dot{\phi} + \dot{D}\dot{\phi}\left(\ln\frac{a}{a_0} + \frac{d\ln V_D}{dD}\right) \simeq -V'(\phi), \tag{97}$$

where V_D is the space-like section for $k = 0$ and so

$$\frac{d\ln V_D}{dD} = \ln\chi_c + \frac{1}{2}\ln\pi - \frac{1}{2}\psi\left(\frac{D}{2} + 1\right). \tag{98}$$

Here Euler's psi function ψ is the logarithmic derivative of the gamma function, $\psi(x) \equiv \Gamma'(x)/\Gamma(x)$. Substituting the dimensional constant (83) into Eq. (96), the dynamics of the space dimension is given by

$$\dot{D}^2 \simeq \frac{16\pi D^3 V(\phi)}{C^2(D-1)M_P^2}. \tag{99}$$

Using Eqs. (96) and (97), one finds

$$\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{\epsilon}{D\left(\frac{1}{D_0} - \frac{1}{C}\frac{d\ln V_D}{dD}\right)} - \eta, \tag{100}$$

where the slow-roll parameter in the variable space dimension are defined by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{(D-1)M_P^2}{32\pi D\left(\frac{1}{D_0} - \frac{1}{C}\frac{d\ln V_D}{dD}\right)}\left(\frac{V'}{V}\right)^2 + \frac{D(1-2D)}{2C(D-1)}, \tag{101}$$

$$\eta \equiv \frac{(D-1)M_P^2}{16\pi D\left(\frac{1}{D_0} - \frac{1}{C}\frac{d\ln V_D}{dD}\right)}\left(\frac{V''}{V}\right) + \frac{\frac{-2}{C} + \frac{D^2}{C^2}\frac{d^2\ln V_D}{dD^2} + \frac{\ddot{D}}{H^2D^2}\left(\ln\frac{a}{a_0} + \frac{d\ln V_D}{dD}\right)}{D\left(\frac{1}{D_0} - \frac{1}{C}\frac{d\ln V_D}{dD}\right)}. \tag{102}$$

Note that in the limit of $C \to +\infty$, these slow-roll parameters approach Eqs. (70) and (71), respectively.

B. Values of a_0, Ω_0, C and δ

Taking $k = 0$ in Eq. (88), one gets the critical density [3]

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[3]Substituting $k = 0$ and $\rho = \rho_C$ in Eq. (60), one gets the same expression for ρ_C as when the space dimensions are constant. So, the definition of ρ_C is the same regardless of whether the space dimensions are constant. In three-space, Eq. (103) approaches Eq. (21).

$$\rho_C \equiv \frac{H^2 D(D-1) M_P^2}{16\pi}. \tag{103}$$

From Eqs. (88) and (103), one can get for the present time

$$\Omega_0 = 1 + \frac{k}{a_0^2 H_0^2}, \tag{104}$$

where $\Omega_0 \equiv \Omega_{0m} + \Omega_{0\Lambda}$. Due to present observations [41], we adopt the ΛCDM model with $\Omega_0 = 1$ and cold non-baryonic dark matter with negligible interaction. The CMB constraints, when combined with observations of distant type Ia supernova, are converging on a Λ dominated Universe with $\Omega_{0m} \simeq 0.3$ and $\Omega_{0\Lambda} \simeq 0.7$ [42].

According to Eq. (104), it is easily seen that for a specially flat Universe, i.e. $\Omega_0 = 1$ and $k = 0$, the dependence of a_0 and H_0^{-1} is not determined. Instead, for $k = -1, +1$, corresponding to $\Omega_0 < 1$ and $\Omega_0 > 1$, respectively, the value of a_0 strongly depends on the value of H_0^{-1}. For an open Universe, Eq. (104) leads to $a_0 > H_0^{-1}$. Our treatments here are based on an open Universe with $\Omega_{0\Lambda} = 0$ and $\Omega_0 = \Omega_{0m} = 0.3$. Substituting $k = -1$ and $\Omega_0 = 0.3$ into Eq. (104), we obtain

$$a_0 = 1.106 \times 10^{28} h_0^{-1}\text{cm} = 5.60 \times 10^{41} h_0^{-1}\text{GeV}^{-1}. \tag{105}$$

Most measurements are now consistent with a value $h = 0.65 \pm 0.08$ in units of 100 km sec^{-1} Mpc^{-1} [43]. For simplicity, here we take $h = 1$. Using Eqs. (81) and (105), one finds the value of C and δ for $D_P = 4, 10, 25$, and $+\infty$ with $D_0 = 3$, see Table II. Comparing the values of C and δ in Table I with those of in Table II, one gets that there are a few differences between them. Our following results are based on the values of a_0 and C given in Table II. However our results in the next sections are approximately correct even if one considers the values of C and δ given in our previous works [1,2].

C. Chaotic inflation with variable space dimension

In variable space dimensions, the general potential for chaotic inflation can be taken as $V \propto \phi^\alpha$. In higher dimensions, the value of α should be specified so that the theory is renormalisable. We are interested now in studying the simplest chaotic inflation with the $m^2\phi^2/2$ potential in the model Universe with variable space dimensions. We begin by calculating the number of e-foldings. To do this, one needs the value of $|1 - \Omega_f^{-1}|$. Using Eqs. (88) and (103), we are led to

$$|1 - \Omega_f^{-1}| = \frac{D_f(D_f - 1) M_P^2 |k|}{16\pi \rho_f a_f^2}, \tag{106}$$

where the subscript f denotes the value at the GUT epoch, i.e. $\rho_f \equiv \rho_{GUT}$, $a_f \equiv a_{GUT}$, $D_f \equiv D_{GUT}$ and $|1 - \Omega_f^{-1}| \equiv |1 - \Omega_{GUT}^{-1}|$. At some time $t = t_{eq}$ in the past (corresponding to a value $a = a_{eq}$ and redshift $z = z_{eq}$) the radiation and matter will have equal density and we have [12]

$$(1 + z_{eq}) = \frac{a_0}{a_{eq}} \simeq 3.9 \times 10^4 (\Omega\, h^2). \tag{107}$$

Once the temperature of the radiation grows as a^{-1}, the temperature of the Universe at this epoch will be

$$T_{eq} = T_{now}(1 + z_{eq}) = 9.24\, (\Omega\, h^2)\, \text{eV}. \tag{108}$$

Using $T_f \simeq 10^{24}\, (\Omega\, h^2)\, \text{eV}$ and thanks to the fact that $(a_f/a_{eq}) = (T_{eq}/T_f)$, we obtain $a_f \simeq 2.37 \times 10^{-28} a_0\, (\Omega h^2)$. So, from Eq. (105), one gets $a_f \simeq 2.62$cm. Substituting this value into the dimensional constraint (82), the corresponding value of D_f can be obtained. The space dimension $D = D_{eq}$, when $a = a_{eq}$, can be obtained by Eqs. (82) and (107), see Table II. From Eq. (93), the energy density at the time $t = t_{eq}$ is given by

$$\rho_{eq} = \rho_0 e^{C(D_{eq}/D_0 - 1)} \frac{V_{D_0}}{V_{D_{eq}}}. \tag{109}$$

Substituting this into Eq. (92), one gets the energy density at the GUT epoch

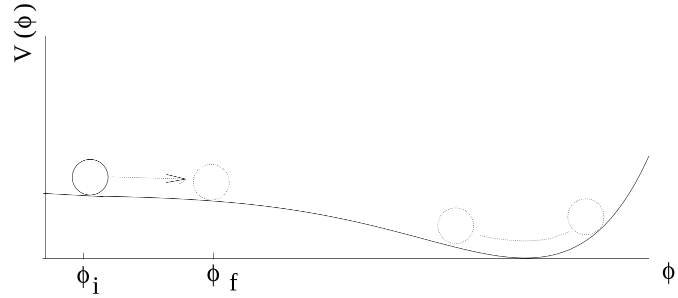

FIG. 1. Schematic illustration of an inflationary potential. Notice the "flatness" of the potential, a feature common to all inflationary models. For the slow-roll approximation, $V(\phi)$ is nearly constant between ϕ_i and ϕ_f and therefore $\rho_i \simeq \rho_f$.

$$\rho_f = \rho_0 e^{C(D_{eq}/D_0-1)} \frac{V_{D_0}}{V_{D_{eq}}} e^{C(D_f/D_{eq}-1)} \left(\frac{D_f}{D_{eq}}\right)^{C/D_{eq}} \frac{V_{D_{eq}}}{V_{D_f}} \exp\left(-\int_{D_{eq}}^{D_f} dD \frac{1}{D} \frac{d\ln V_D}{dD}\right). \tag{110}$$

Considering an open Universe with $\Omega_0 = \Omega_{0m} = 0.3$ and $\Omega_{0\Lambda} = 0$, we have

$$\rho_0 \equiv \Omega_0\, \rho_c \simeq 5.64 \times 10^{-30} h^2 \text{ gr cm}^{-3} \simeq 2.43 \times 10^{-47} h^2 \text{GeV}^4.$$

It should be noted that the value of space-like sections in Eq. (110) is for $k = -1$. So, from Eq. (86) one gets

$$\int_{D_{eq}}^{D_f} dD \frac{1}{D} \frac{d\ln V_D}{dD} = \frac{1}{2} \ln\pi \ln \frac{D_f}{D_{eq}} - \frac{1}{2} \int_{D_{eq}}^{D_f} dD \frac{1}{D} \psi\left(\frac{D}{2}\right), \tag{111}$$

where we use $\frac{d\ln f(\chi_c)}{dD} = 0$. From Eqs. (106), (110) and (111), one finds the value of $|1 - \Omega_f^{-1}|$, see Table II [4].

To calculate the number of e-foldings $\mathcal{N} \equiv \ln(a_f/a_i)$, we use Eq. (106) for some finite time interval $t \in [t_i, t_f]$ of the inflationary epoch

$$\frac{\rho_i a_i^2 |1 - \Omega_i^{-1}|}{D_i(D_i - 1)} = \frac{\rho_f a_f^2 |1 - \Omega_f^{-1}|}{D_f(D_f - 1)}. \tag{112}$$

In Fig. 1, the inflaton is represented by a little ball that rolls down the potential hill. Near the top of the potential the slope is very small, so the roll is slow and $V(\phi) \simeq$ const.. In other words $\rho_i \simeq \rho_f$. Considering $\rho_i \simeq \rho_f$ and $\Omega_i \approx \mathcal{O}(1)$ and using the dimensional constraint (82), one can write down Eq. (112) as

$$\left(\frac{a_f}{a_i}\right)^2 \left(\frac{1}{\frac{1}{C}\ln\frac{a_f}{a_i} - \frac{1}{D_f}}\right) \left(\frac{1}{\frac{1}{C}\ln\frac{a_f}{a_i} - \frac{1}{D_f}} + 1\right) = \frac{D_f(D_f - 1)}{|1 - \Omega_f^{-1}|}. \tag{113}$$

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[4]It should be noted that in the limit of constant space dimension, we have

$$\lim_{C\to+\infty} \left(\frac{D}{D_{eq}}\right)^{C/D_{eq}} = \lim_{C\to+\infty} \left(\frac{D}{C} \ln \frac{a_{eq}}{a} + 1\right)^{C/D_{eq}} = \frac{a_{eq}}{a},$$

and also

$$\lim_{C\to+\infty} e^{C(D/D_{eq}-1)} = \left(\frac{a_{eq}}{a}\right)^{D_0}.$$

So, from Eqs. (106) and (110), one gets in the constant three-space

$$|1 - \Omega_f^{-1}| = \frac{3M_P^2}{8\pi\rho_{eq}\,(a_{eq}/a_f)^4\, a_f^2} \simeq 5.113 \times 10^{-51}.$$

TABLE II. Values of C, $\log(\delta/l_P)$, δ, D_f, D_{eq}, $|1-\Omega_f^{-1}|$, $\mathcal{N}$, D_i, a_i, ϕ_f/M_P and ϕ_i/M_p. We take $D_0 = 3$ and $a_0 \simeq 1.106 \times 10^{28} h_0^{-1}$cm. Since for $D_P \geq 10$ the value of D_i is negative, the values of ϕ_f/M_P and ϕ_i/M_p are not calculated and are denoted by a star.

D_P	C	$\log_{10}(\delta/l_P)$	δ(cm)	D_f	D_{eq}	$\lvert 1-\Omega_f^{-1}\rvert$	$\mathcal{N}$	D_i	a_i(cm)	ϕ_f/M_P	ϕ_i/M_P
3	$+\infty$	$-\infty$	0	3.00	3.00	5.113×10^{-51}	57.9	3.00	1.87×10^{-25}	$1/\sqrt{4\pi}$	3.049
4	1680.93	-182.50	5.1×10^{-216}	3.38	3.06	7.832×10^{-61}	69.2	3.93	2.32×10^{-30}	0.290	3.474
10	600.33	-26.07	1.4×10^{-59}	4.40	3.17	1.544×10^{-86}	97.4	15.35	1.31×10^{-42}	0.304	4.464
25	477.54	-8.30	8.2×10^{-42}	5.00	3.21	1.324×10^{-101}	114.3	-25.50	6.00×10^{-50}	$\star$	$\star$
$+\infty$	420.23	0	l_P	5.50	3.24	4.203×10^{-114}	115.3	-10.80	2.21×10^{-50}	$\star$	$\star$

From this equation, we find the value of $\mathcal{N}$ by numerical calculations. The initial size of the Universe a_i at the beginning of inflation is given by $a_i = a_f \exp(-\mathcal{N})$. Using the number of e-folding and Eq. (82), it is easy to calculate the space dimension D_i at the beginning of inflation. The values of $\mathcal{N}$, a_i and D_i are given in Table II.

It is worth mentioning that for $D_P \geq 25$, D_i takes a negative value. Also for $D_P = 10$, D_i exceeds the value of D_P; in other words a_i is smaller than the Planck length. This means that the inflationary period of the history of the Universe with variable space dimensions and $D_P = 10$ takes place before the Planck epoch. This is even more surprising when one realizes that during inflation a region of initial size is $\Delta l \sim l_P \sim 10^{-33}$cm. Therefore we rule out the cases with $D_P \geq 10$ in the model Universe with variable space dimensions. In what follows, we will be particularly interested in the case $D_P = 4$ and $C = 1680.93$ for which inflation begins from an initial size of about $a_i \sim 10^{-30}$cm $\sim 10^3 \times l_P$ and with an initial space dimension $D_i \simeq 3.93$.

Let us now describe how to obtain the initial and the final value of the inflaton field denoted by ϕ_i and ϕ_f, respectively. The end of inflation is marked by the failure of one of three slow-roll conditions, see Eq. (76). Taking $\epsilon_f = \eta_f = 1$, one can use Eqs. (101) and (102) to calculate ϕ_f/M_P

$$\frac{\phi_f}{M_p} = \sqrt{\frac{D_f - 1}{8\pi D_f \left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi + \frac{1}{2C}\psi\left(\frac{D_f}{2}+1\right)\right]}}. \tag{114}$$

It should be noted that we neglect the second term on the RHS of Eqs. (101) and (102), because these terms are independent of the shape of the inflaton potential and make a small contribution to the value of ϵ and η. Using Eq. (114), one finds $\phi_f/M_P = 0.290$ for $D_P = 4$, see Table II. Notice the value of ϕ_f/M_P depends on χ_C. For simplicity, we take $\chi_c = 1$ and ignore its contribution to the value of ϕ_f/M_P. For more details about the value of χ_C, see the Appendix. Using Eqs. (83), (96)-(98), it is easily shown that

$$\left(\frac{\phi}{M_P}\right)^2 = \frac{C}{4\pi}\int_{D_i}^{D} \frac{dD'(D'-1)}{D'^3\left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_C - \frac{1}{2C}\ln\pi + \frac{1}{2C}\psi\left(\frac{D'}{2}+1\right)\right]} + \left(\frac{\phi_i}{M_P}\right)^2. \tag{115}$$

For simplicity and using a fine approximation, one can neglect the term including the Euler's psi function, which is of the order of $1/C$, compared to the term of order $\frac{1}{D_0}$. Therefore, Eq. (115) can be written as

$$\left(\frac{\phi}{M_P}\right)^2 = \frac{C\left[\frac{1}{D_i}\left(1-\frac{1}{2D_i}\right) - \frac{1}{D}\left(1-\frac{1}{2D}\right)\right]}{4\pi\left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi\right]} + \left(\frac{\phi_i}{M_p}\right)^2. \tag{116}$$

Substituting $\phi = \phi_f$, $D = D_f$ and the corresponding values of C and D into Eq. (116), one gets $\phi_i/M_P = 3.474$, see Table II.

Using Eqs. (82) and (116), one finds a relation between $\phi(t)$ and $a(t)$ which is given by

$$\left(\frac{\phi}{M_P}\right)^2 = \frac{\left(-1 + \frac{1}{D_i} + \frac{1}{2C}\ln\frac{a}{a_i}\right)\ln\frac{a}{a_i}}{4\pi\left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi\right]} + \left(\frac{\phi_i}{M_p}\right)^2. \tag{117}$$

Fig. 2 and Fig. 3 show $\phi(t)$ as a function of $\ln(a/a_i)$ and $D(t)$, respectively, during the inflationary scenario, $t \in [t_i, t_f]$.

Let us now obtain the evolution equations of $\phi(t)$, $a(t)$ and $D(t)$. Solving Eq. (116), one can get D in terms of

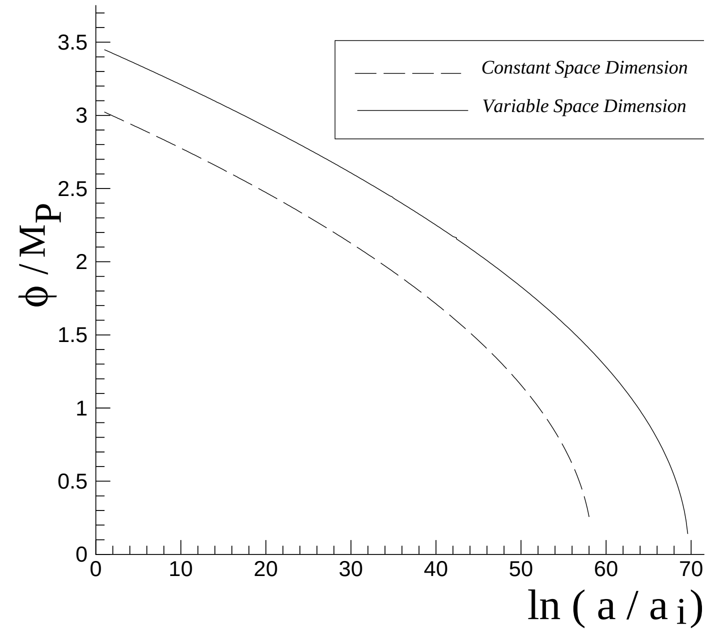

FIG. 2. $\phi(t)/M_P$ as a function of $\ln(a/a_i)$ during the inflationary epoch. The dashed line is for three-space with $a_i \simeq 1.87 \times 10^{-25}$cm and the solid line is for the case of variable space dimension with $D_P = 4$ and $a_i \simeq 2.32 \times 10^{-30}$cm.

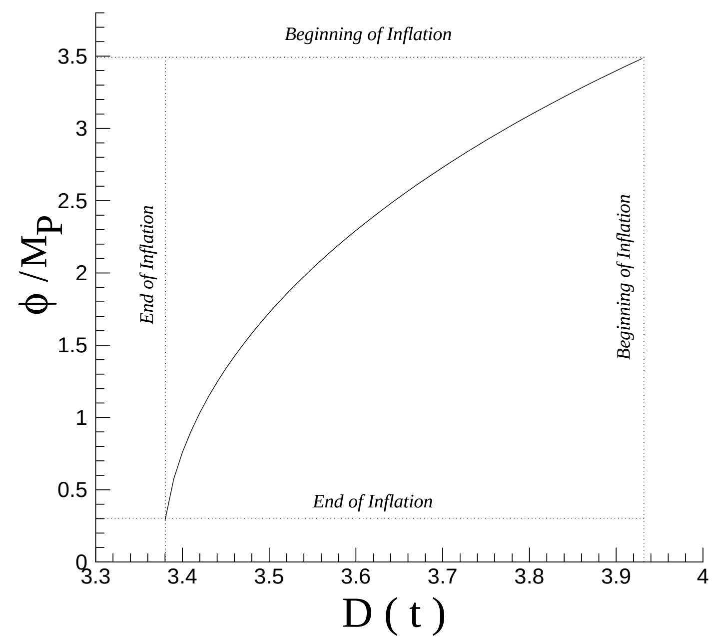

FIG. 3. $\phi(t)/M_P$ as a function of $D(t)$ for $D_P = 4$ during the inflationary epoch.

ϕ/M_P. This relation is given by [5]

$$D(\phi) = \frac{1 + \sqrt{1 - \frac{2}{D_i}\left(1 - \frac{1}{2D_i}\right) - \frac{8\pi}{C}\left(\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi\right)\left[(\phi_i/M_P)^2 - (\phi/M_P)^2\right]}}{2\left\{\frac{1}{D_i}\left(1 - \frac{1}{2D_i}\right) + \frac{4\pi}{C}\left(\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi\right)\left[(\phi_i/M_P)^2 - (\phi/M_P)^2\right]\right\}}. \tag{118}$$

Using Eqs. (83), (96) and (97), the classical equation of motion for ϕ is given by

$$\dot{\phi} = -\frac{mM_P\sqrt{\frac{D-1}{8\pi D}}}{D\left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_C - \frac{1}{2C}\ln\pi + \frac{1}{2C}\psi\left(\frac{D}{2}+1\right)\right]}. \tag{119}$$

This just tells us that the inflaton field is like a ball rolling down a hill, $\dot{\phi} < 0$. Substituting D from Eq. (118) into (119), we are led to the non-linear differential equation $\dot{\phi} \equiv f(\phi)$. We approximate this differential equation by expanding $f(\phi)$ in inverse powers of C. As mentioned previously, we take $C = 1680.93$ corresponding to $D_P = 4$, $\phi_i/M_P = 3.474$ and $D_i = 3.93$. We are therefore led to

$$\dot{\phi} = -\frac{0.23mM_p}{\sqrt{\pi}} - \frac{mM_P}{C\sqrt{\pi}}\left\{1.3\pi\left[\left(\frac{\phi_i}{M_P}\right)^2 - \left(\frac{\phi}{M_P}\right)^2\right] + 0.687\left(\ln\chi_c + \frac{\ln\pi}{2} - 0.45\right)\right\} + \mathcal{O}\left(\frac{1}{C^2}\right) + ... \tag{120}$$

The above differential equation is the so-called Riccati's equation [44]. It can be shown that the solution of (120) is given by

$$\begin{aligned}\phi(t) = \phi_i &- \frac{0.23mM_P}{\sqrt{\pi}}t \\ &- \frac{M_P}{C\sqrt{\pi}}\left\{-0.023m^3t^3 + 0.300\sqrt{\pi}m^2t^2\frac{\phi_i}{M_P} + 0.687mt\left(\ln\chi_c + \frac{\ln\pi}{2} - 0.45\right)\right\} + \mathcal{O}\left(\frac{1}{C^2}\right) + ...\end{aligned} \tag{121}$$

To obtain the dynamical behavior of the scale factor, we write down Eq. (96) as

$$\frac{\dot{a}}{a} = \frac{2m\phi}{M_P}\sqrt{\frac{2\pi}{D(D-1)}}. \tag{122}$$

It is difficult to solve this differential equation because D is a function of ϕ, see Eq. (118). From Eqs. (118), (121), and (122) one can write down $\dot{a}/a \equiv g(t)$, where $g(t)$ is a non-linear function of time. By expanding $g(t)$ in inverse powers of C, we are led to

$$\begin{aligned}\frac{\dot{a}}{a} &= 0.834m\sqrt{\pi}\left(\frac{\phi_i}{M_P} - 0.23\frac{mt}{\sqrt{\pi}}\right) \\ &+ \frac{1}{C}\left\{6.858m\pi^{3/2}\left(\frac{\phi_i}{M_P} - 0.23\frac{mt}{\sqrt{\pi}}\right)\left[\left(\frac{\phi_i}{M_P}\right)^2 - \left(\frac{\phi_i}{M_P} - 0.23\frac{mt}{\sqrt{\pi}}\right)^2\right]\right. \\ &\left. - 0.834m\left[-0.023m^3t^3 + 0.300\sqrt{\pi}m^2t^2\frac{\phi_i}{M_P} + 0.687mt\left(\ln\chi_c + \frac{\ln\pi}{2} - 0.45\right)\right]\right\} \\ &+ \mathcal{O}\left(\frac{1}{C^2}\right) + ...\end{aligned} \tag{123}$$

Integrating this equation one gets

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[5] It is worth mentioning that Eq. (118) approaches $D = D_0$ in the limit of constant space dimension or $C \to +\infty$.

$$a(t) = a_i \exp\Bigg\{1.813\pi\left[\left(\frac{\phi_i}{M_P}\right)^2 - \left(\frac{\phi_i}{M_P} - \frac{0.23mt}{\sqrt{\pi}}\right)^2\right]\left(1 + 8.223\frac{\pi}{C}\left(\frac{\phi_i}{M_P}\right)^2\right)$$
$$+ \frac{1}{C}\Bigg(7.454\pi^2\left[\frac{\phi_i}{M_P} - \frac{0.23mt}{\sqrt{\pi}}\right]^4 + 0.834m^2\Bigg[0.006m^2t^4 - 0.100\sqrt{\pi}mt^3\frac{\phi_i}{M_P}$$
$$- 0.344\left(\ln\chi_c + \frac{\ln\pi}{2} - 0.45\right)\Bigg]\Bigg)\Bigg\} + \mathcal{O}\left(\frac{1}{C^2}\right) + ... \qquad (124)$$

Substituting the above equation into the constraint (82) leads to

$$\frac{1}{D(t)} = \frac{1}{D_i} + \frac{1}{C}\left\{1.813\pi\left[\left(\frac{\phi_i}{M_P}\right)^2 - \left(\frac{\phi_i}{M_P} - \frac{0.23mt}{\sqrt{\pi}}\right)^2\right]\right\}$$
$$+ \mathcal{O}\left(\frac{1}{C}\right)^2 + ... \qquad (125)$$

In Sec. II, we determined the spectrum and the spectral index in the constant three-space. We are now interested in the value of observational quantities n, n_T, and r in the model Universe with variable space dimensions. The numerical coefficients in Eqs. (39) and (40) are for three-space. These coefficients will depend on the space dimension. To obtain their dependence on the space dimension one has to generalize the formalism given in Ref. [45] to the case of variable space dimensions. In three-space, the numerical coefficient in Eq. (39) was first given by Lyth [45]. We generally use these coefficients even in the case of variable space dimensions. The variability of the space dimensions affects our results by changing the slow-roll parameters in the model Universe with variable space dimensions as given in Eqs. (101) and (102).

To evaluate $\phi_\star$ when the scale $k_\star$ leaves the horizon, *i.e.* $k_\star = aH$, we generalize Eq.(45) to the case of variable space dimensions. Using Eqs. (96) and (97), one can write down Eq. (45) as

$$\mathcal{N}(\phi) = -\frac{16\pi}{M_P^2}\int_\phi^{\phi_f}\frac{D}{(D-1)}\left[\frac{1}{D_0} - \frac{1}{C}\ln\chi_c - \frac{1}{2C}\ln\pi + \frac{1}{2C}\psi\left(\frac{D}{2}+1\right)\right]\frac{V}{V'}\,d\phi. \qquad (126)$$

Taking $\mathcal{N}_\star = 60$ for the $m^2\phi^2/2$ potential, $D_P = 4$, and $C = 1680.93$, one gets $\phi_\star = 3.23\,M_P$, $\epsilon_\star = 6.661\times10^{-3}$, and $\eta_\star = 8.475\times10^{-3}$. Using Eqs. (39)-(42), (44), (47), and (48) one finds

$$A_S(k_\star) = 17.082\,m/M_P \;\;,\;\; A_T(k_\star) = 1.492\,m/M_P, \qquad (127)$$
$$n(k_\star) = 0.977 \;\;,\;\; n_T(k_\star) = -1.322\times10^{-2} \;\;,\;\; r(k_\star) = 8.196\times10^{-2}, \qquad (128)$$
$$\left.\frac{dn(k)}{d\ln k}\right|_{k_\star} \simeq -1.52\times10^{-4} \;\;,\;\; \left.\frac{dn_T(k)}{d\ln k}\right|_{k_\star} \simeq -1.25\times10^{-4}. \qquad (129)$$

These values can be compared with our results as given by Eqs. (49), (50), (55), and (56). From the value of $A_S(k_\star) = 17.082\,m/M_p$ and the COBE renormalization $\delta_H \simeq 1.91\times10^{-5}$, one can get $m \simeq 1.12\times10^{-6}M_P$. Moreover the inflationary energy scale is given by

$$V_\star^{1/4} = 6.60{\epsilon_\star}^{1/4}\times10^{16}\mathrm{GeV} \simeq 1.88\times10^{16}\mathrm{GeV}. \qquad (130)$$

We can approximately obtain t_f from Eq. (121)

$$t_f \simeq \frac{(\phi_i - \phi_f)\sqrt{\pi}}{0.23mM_P} \sim 1.18\times10^{-36}\mathrm{sec}. \qquad (131)$$

At this time the Universe expands to a huge size

$$a(t) \sim a_i\exp\left(1.813\pi\frac{\phi_i^2}{M_P^2}\right) \simeq 10^{-30}\exp\left(\frac{1.813\pi M_P^2}{m^2}\right) > 10^{10^{12}}\mathrm{cm}, \qquad (132)$$

where we use $\phi_i \lesssim M_P^2/m$.

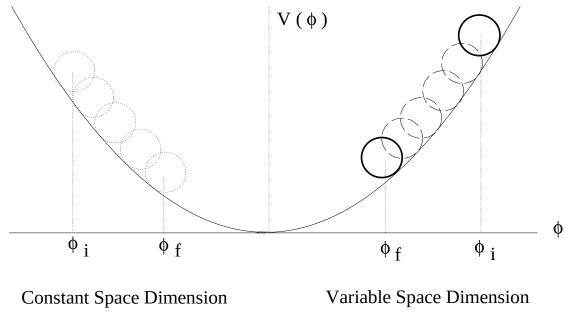

FIG. 4. Generic form of a shift in the values of ϕ_i and ϕ_f. In three-space $\phi_i \simeq 3.049 M_P$ and $\phi_f \simeq M_P/\sqrt{4\pi}$. In the case of variable space dimension with $D_P = 4$, $\phi_i \simeq 3.474 M_P$ and $\phi_f \simeq 0.029 M_P$. The dynamical character of the space dimensions shifts the values of ϕ_i and ϕ_f to slightly larger values, producing delayed chaotic inflation.

VI. CONCLUDING REMARKS

In this paper, we study a simple model of chaotic inflation assuming the space dimension as a dynamical parameter. For both constant and also variable space dimension, at the end of inflationary epoch, we assume the Universe would have grown to the size of a ping-pong ball. This is based on cosmological data and we describe how to derive it. It is easily shown that our results are not very sensitive to the size of the Universe at the end of inflation.

In the model, there is a free parameter, called C. For different values of C corresponding to $D_P = 4, 10, 25$ and ∞, we calculate the e-folding number. Our study is based on an open Universe. There is a few differences between the values of C based on an open Universe and those given in [1]. This means that our results are not very sensitive to the value of Ω_0. To obtain the number of e-foldings $\mathcal{N}$, we repeat the analysis of the inflationary cosmology in three-space. For $D_P = 4, 10, 25$ and ∞ we get $\mathcal{N} \sim 70, 98, 114$ and 115, respectively. The dynamical character of the space dimension increases the number of e-foldings (for constant three-space, we have 58 e-foldings). Using these e-foldings, we obtain the initial and final value of the inflaton field.

As illustrated in Fig. 4, there is a small shift in the value of ϕ_i and ϕ_f to larger values. In three-space, inflation ends at $\sim 10^{-37}$sec. We show the end of inflation is $\sim 10^{-36}$sec by assuming the space dimension is a dynamical parameter. Therefore, we conclude that there is a delayed chaotic inflation with variable space dimension.

Using the number of e-foldings, we also obtain the size of the Universe at the beginning of inflationary epoch. For $D_P = 10, 25$ and ∞, inflation starts from an initial size less than the Planck length. We know that in three-space the initial size of the inflationary epoch is of the order of the Planck length. Particularly, for $D_P = 25$ and ∞ the initial size is less than the minimum size of the model, which is δ. So, we rule out the cases of $D_P \geq 10$ in the model and conclude an upper limit for the space dimension at the Planck length, $D_P < 10$. Our result for an upper limit for the space dimension at the Planck length is in agreement with previous result in [2]. Here, we take $D_P = 4$ corresponding to $C = 1680.93$. For this large value of C, we expand the field equations and give the dynamical behavior of the inflaton field, scale factor, and space dimension.

We also estimate the spectral indices and the mass of the inflaton field and other observable quantities in chaotic inflation with variable space dimension. This numerical resuls can be compared with recent Boomerang data.

ACKNOWLEDGMENTS

The authors would like to thank Reza Mansouri, Uroš Seljak, and Edward Kolb for useful comments and discussion, and Michael H. Kesden for correcting several errors in early drafts.

APPENDIX: VALUE OF χ_C

In calculation of the volume of space-like sections for open and flat Universe there is an upper cut-off scale χ_C, see Ref. [1]. In this paper, we assumed $\chi_C = 1$ in comoving coordinates and neglected its contribution in our calculations. Actually χ_C is very large or infinite. To avoid any infinite values for observational quantities, we need to specify a

finite value of χ_C in physical coordinates. Here we point out three choices for the value of χ_C:
i) The present size of the horizon $\sim 3000h^{-1}Mpc$,
ii) A modest value of the cross-over scale from fractal distribution of the matter to homogeneity $\sim 100h^{-1}Mpc$,
iii) The causal horizon, given by the physical Hubble length, which is constant during the inflationary epoch, *i.e.* $H^{-1} \sim l_P$.

To study the effect of dynamical character of space dimension in the inflationary epoch, one may choose $\chi_C \sim l_P$. The values of $|1-\Omega_f^{-1}|$, $\mathcal{N}$, D_i, D_f and a_i do not depend on the value of χ_C. Taking $\chi_C \simeq l_P$, for $D_P = 4$ corresponding to $C = 1680.93$, we find the following values:

$$\begin{aligned}
&\phi_f = 0.272 M_P \ , \ \phi_i = 3.260 M_P, \\
&\epsilon_\star = 5.828 \times 10^{-3} \ , \ \eta_\star = 8.488 \times 10^{-3} \ , \ \phi_\star = 3.03 M_P, \\
&A_S^2(k_\star) = 17.082 m/M_P \ , \ A_T^2(k_\star) = 1.492 m/M_P, \\
&n(k_\star) = 0.982 \ , \ n_T(k_\star) = -1.166 \times 10^{-2} \ , \ r(k_\star) = 7.227 \times 10^{-2}, \\
&\left.\frac{dn}{d\ln k}\right|_{k_\star} = -2.37 \times 10^{-5} \ , \ \left.\frac{dn_T}{d\ln k}\right|_{k_\star} = -7.39 \times 10^{-5}, \\
&V_\star^{1/4} = 1.82 \times 10^{16} GeV \ , \ m \simeq 1.12 \times 10^{-6} M_P.
\end{aligned}$$

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